Comparing Methods for Supersonic MHD Turbulence

Alexei Kritsuk

University of California, San Diego

Collaborators:

Åke Nordlund (NBI Copenhagen, DK),
Paolo Padoan (ICREA/ICC, Barcelona, ES),
Tom Abel (KIPAC & SLAC, Stanford, CA),
Cristoph Federrath (ENS Lyon, FR),
Dongwook Lee (U Chicago, IL),
Wolf-Christian Müller (IPP, Garching, DE),
Sergey Ustyugov (Keldysh, Moscow, RU),
Hao Xu (LANL, Los Alamos, NM)

David Collins (LANL, Los Alamos, NM),
Mike Norman (SDSC, La Jolla, CA),
Robi Banerjee (Hamburg Obs., DE),
Mario Flock (MPIA, Heidelberg, DE),
Pak Shing Li (UC Berkeley, CA),
Romain Teyssier (ITP Zurich, CH),
Christian Vogel (IPP, Garching, DE),
Motivation

Towards *ab initio* simulations of star formation in molecular clouds (MCs):

- Supersonic turbulence plays an important role in fragmentation of MCs leading to star formation.

- Numerical simulations of highly compressible MHD turbulence is a challenging problem (accuracy & stability):
  - Mach numbers: $M_S \sim 10, \; M_A \geq 1$.
  - Reynolds numbers: $Re \sim 10^8, \; Rm \sim 10^{13}$.
  - Magnetic Prandtl number: $Pm \sim 10^5$.
  - In simulations $Pm \sim 1$; critical issue for the small-scale dynamo action.

- Need to better understand the quality of MHD solvers we use to model star formation in the turbulent ISM.
High-order accurate numerical methods for compressible MHD turbulence.

Numerical stability is often an issue in supersonic regime.

ILES approach is often used, but numerical dissipation is poorly understood.

Nine MHD methods (all 9 are grid-based schemes, SPMHD – withdrawn).

ENZO, FLASH, KT-MHD, LL-MHD, PLUTO, PPML, RAMSES, STAGGER, and ZEUS.

Isothermal turbulence decay problem from initial conditions generated with the STAGGER code.

Initial rms sonic Mach number, $M_s = 9$; Alfvénic Mach number, $M_A = 4.5$.

Grid resolution: $256^3$ and $512^3$ (also $1024^3$ for PPML and ZEUS).

Decay is followed for 4 dynamical times, $t \in [0, 0.2]$. 

COMPARING NUMERICAL METHODS FOR ISOTHERMAL MAGNETIZED SUPersonic TURBULENCE

ALEXEI G. KRITSUK,1,2 ÅKE NORDLUND,2,3 DAVID COLLINS,1,2,4 PAOLO PAOIAN,2,5 MICHAEL L. NORMAN,1,6 TOM ABEL,2,7 ROBI BANERJEE,2,8,9 CHRISTOPH FEDERRATH,8,10,11 MARIO FLOCK,10 DONGWOOK LEE,12 PAK SHING LI,2,13 WOLF-CHRISTIAN MÜLLER,14 ROMAIN TEYSSIER2,15,16 SERGEY D. USTYUGOV,17 CHRISTIAN VOGEL,14 AND HAO XU1,4

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ABSTRACT

Many astrophysical applications involve magnetized turbulent flows with shock waves. Ab initio star formation simulations require a robust representation of supersonic turbulence in molecular clouds on a wide range of scales imposing stringent demands on the quality of numerical algorithms. We employ simulations of supersonic super-Alfvénic turbulence decay as a benchmark test problem to assess and compare the performance of nine popular astrophysical MHD methods actively used to model star formation. The set of nine codes includes: ENZO, FLASH, KT-MHD, LL-MHD, PLUTO, PPML, RAMSES, STAGGER, and ZEUS. These applications employ a variety of numerical approaches, including both split and unsplit, finite difference and finite volume, divergence preserving and divergence cleaning, a variety of Riemann solvers, a range of spatial reconstruction and time integration techniques. We present a comprehensive set of statistical measures designed to quantify the effects of numerical dissipation in these MHD solvers. We compare power spectra for basic fields to determine the effective spectral bandwidth of the methods and rank them based on their relative effective Reynolds numbers. We also compare numerical dissipation for solenoidal and dilatational velocity components to check for possible impacts of the numerics on small-scale density statistics. Finally, we discuss convergence of various characteristics for the turbulence decay test and impacts of various components of numerical schemes on the accuracy of solutions. The nine codes gave qualitatively the same results, implying that they are all performing reasonably well and are useful for scientific applications. We show that the best performing codes employ a consistently high order of accuracy for spatial reconstruction of the evolved fields, transverse gradient interpolation, conservation law update step, and Lorentz force computation. The best results are achieved with divergence-free evolution of the magnetic field using the constrained transport method, and using little to no explicit artificial viscosity. Codes which fall short in one or more of these areas are still useful, but they must compensate higher numerical dissipation with higher numerical resolution. This paper is the largest, most comprehensive MHD code comparison on an application-like test problem to date. We hope this work will help developers improve their numerical algorithms while helping users to make informed choices in picking optimal applications for their specific astrophysical problems.

Subject headings: ISM: structure — Magnetohydrodynamics: MHD — methods: numerical — turbulence
Conservation laws

Ideal compressible isothermal MHD;

mass, momentum, and flux conservation:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{1}
\]

\[
\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot \left[ \rho \mathbf{u} \mathbf{u} - \mathbf{B} \mathbf{B} + \left( p + \frac{\mathbf{B}^2}{2} \right) \mathbf{I} \right] = \mathbf{F}, \tag{2}
\]

\[
\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{u} \mathbf{B} - \mathbf{B} \mathbf{u}) = 0. \tag{3}
\]

- Pressure: \( p = c_s^2 \rho \); sound speed: \( c_s \equiv 1 \).
- Solenoidal constraint on the \( \mathbf{B} \)-field: \( \nabla \cdot \mathbf{B} \equiv 0 \).
- Gravity is not included.
- Forcing: \( \mathbf{F} \equiv \rho \mathbf{a} - \langle \rho \mathbf{a} \rangle \); large-scale solenoidal acceleration: \( \mathbf{a}(\mathbf{x}, t) \).
- Initial conditions: \( \rho_0 + \delta \rho, p_0, \mathbf{u}_0 = \tau \mathbf{a}, \mathbf{B}_0 = (0, 0, B_0) \); periodic boundaries.
- Implicit large eddy simulation (ILES) [Sytine et al., 2000; Grinstein et al., 2007].
### Solver design basics

<table>
<thead>
<tr>
<th>Code</th>
<th>BaseMthd(^a)</th>
<th>SptlOrdr(^b)</th>
<th>SrcTerms(^c)</th>
<th>MHD(^d)</th>
<th>TimeIntgr(^e)</th>
<th>DirSplit(^f)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ENZO</td>
<td>FV, HLL</td>
<td>2nd</td>
<td>Dedner</td>
<td>Dedner</td>
<td>2nd-order RK</td>
<td>Direct</td>
</tr>
<tr>
<td>FLASH</td>
<td>FV, HLLD</td>
<td>2nd</td>
<td>(\parallel) Derivative</td>
<td>3rd-order CT</td>
<td>Forward Euler</td>
<td>(\perp) Recon.</td>
</tr>
<tr>
<td>KT-MHD</td>
<td>FD, CWENO</td>
<td>3rd</td>
<td>KT</td>
<td>3rd-order CT</td>
<td>4th-order RK</td>
<td>Direct</td>
</tr>
<tr>
<td>LL-MHD</td>
<td>FV, HLLD</td>
<td>2nd</td>
<td>None</td>
<td>Athena CT</td>
<td>Forward Euler</td>
<td>Split</td>
</tr>
<tr>
<td>PLUTO</td>
<td>FV, HLLD</td>
<td>3rd</td>
<td>Powell</td>
<td>Powell</td>
<td>3rd-order RK</td>
<td>Direct</td>
</tr>
<tr>
<td>PPML</td>
<td>FV, HLLD</td>
<td>3rd</td>
<td>None</td>
<td>Athena CT</td>
<td>Forward Euler</td>
<td>(\perp) Recon.</td>
</tr>
<tr>
<td>RAMSES</td>
<td>FV, HLLD</td>
<td>2nd</td>
<td>None</td>
<td>2D HLLD CT</td>
<td>Forward Euler</td>
<td>(\perp) Recon.</td>
</tr>
<tr>
<td>STAGGER</td>
<td>FD, Stagg.</td>
<td>6th</td>
<td>Tensor</td>
<td>Stagg. CT</td>
<td>3-order Hyman</td>
<td>Direct</td>
</tr>
<tr>
<td>ZEUS</td>
<td>FD, vLeer</td>
<td>2nd</td>
<td>vNeumann</td>
<td>MOC-CT</td>
<td>Forward Euler</td>
<td>Split</td>
</tr>
</tbody>
</table>

\(^a\) Base method. FD for finite-difference, FV for finite-volume. FV techniques have the Riemann solver listed.

\(^b\) Formal spatial order of accuracy.

\(^c\) Artificial Viscosity. \(\parallel\) Derivative” indicates terms \(\propto\) to the longitudinal derivative of B.

\(^d\) MHD method.

\(^e\) Time integration method.

\(^f\) Multidimensional technique. \(\perp\) Reconstruction” indicates transverse derivatives in the interface reconstruction.
Flow fields: a density slice with PPML $(512^3)$
Supersonic MHD turbulence decay

Evolution of the specific kinetic energy density, $512^3$

- All codes agree.
- $E_K$ is determined mostly by the large-scale flow (steep power spectra of velocity, $E_K \propto k^{-1.9}$) ⇒ all codes resolve the large “eddies” sufficiently well.

Supersonic MHD turbulence decay

Evolution of the rms sonic Mach number, $512^3$

- All codes agree, $M_s \sim \sqrt{E_K}$.
- Mach number drops from $\sim 9$ to $\sim 2.5$ as the turbulence decays.
Supersonic MHD turbulence decay

A relative measure of the Reynolds number, $512^3$

- **STAGGER** shows an outstanding result!
- **PPML** and **RAMSES** follow.
- **ZEUS** shows the lowest $Re_{\text{eff}}$.
Compensated velocity power spectra: $512^3$, $t = 0.4t_{\text{dyn}}$

- **STAGGER** again shows an outstanding result.
- **RAMSES**, **PLUTO**, and **PPML** are consistently good.
- **FLASH** and **ZEUS** are the most diffusive.
Compensated velocity power spectra: $512^3$, $t = 1.2 t_{\text{dyn}}$

- **STAGGER** still shows an outstanding result.
- **RAMSES**, **PLUTO**, and **PPML** follow.
- **ENZO**, **KT-MHD**, **FLASH**, and **ZEUS** are the most diffusive.
Compensated velocity power spectra: $512^3$, $t = 4.0 t_{\text{dyn}}$

- **STAGGER** settles down at intermediate-high $k$ (long-term effect of artificial viscosity??).
- **RAMSES**, **PLUTO**, **PPML**, and **LL-MHD** show good results.
- **ENZO**, **KT-MHD**, **FLASH** and **ZEUS** are the most diffusive.
Compensated velocity power spectra: $256^3$, $t = 0.4 \tau_{\text{dyn}}$

- **STAGGER**, **PPML**, and **RAMSES** show the best effective bandwidth.
- Reference solution is $1024^3$ **PPML** data filtered down to $256^3$. 

Supersonic MHD turbulence decay

Dilatational-to-solenoidal ratio, $512^3$, $t = 2.0 t_{\text{dyn}}$

- **KT-MHD** and **ENZO** show substantially higher power in dilatational modes at Nyquist $k$.
- **STAGGER** also shows an excess at intermediate wavenumbers.

Supersonic MHD turbulence decay

Dilatational-to-solenoidal ratio: $512^3$, $t = 4.0 t_{\text{dyn}}$

- Late evolution with KT-MHD, ENZO, and STAGGER is substantially affected in a wide range of intermediate-high wavenumbers.
- Extent of the inertial range may be reduced for KT-MHD, ENZO, and STAGGER.
## Selected numeric values for the velocity field

<table>
<thead>
<tr>
<th>Code</th>
<th>$E_K/E_{K,\text{ref}}$</th>
<th>$2\Omega + 4/3\Delta$</th>
<th>u-bandwidth</th>
<th>$\bar{\chi}(k &gt; 100k_{\text{min}})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ENZO</td>
<td>1.001</td>
<td>0.93</td>
<td>0.19</td>
<td>0.60</td>
</tr>
<tr>
<td>FLASH</td>
<td>1.000</td>
<td>0.85</td>
<td>0.15</td>
<td>0.27</td>
</tr>
<tr>
<td>KT-MHD</td>
<td>1.041</td>
<td>0.89</td>
<td>0.20</td>
<td>0.86</td>
</tr>
<tr>
<td>LL-MHD</td>
<td>1.062</td>
<td>1.02</td>
<td>0.22</td>
<td>0.29</td>
</tr>
<tr>
<td>PLUTO</td>
<td>1.077</td>
<td>1.03</td>
<td>0.20</td>
<td>0.32</td>
</tr>
<tr>
<td>PPML</td>
<td>1.043</td>
<td>1.20</td>
<td>0.24</td>
<td>0.32</td>
</tr>
<tr>
<td>RAMSES</td>
<td>1.069</td>
<td>1.07</td>
<td>0.24</td>
<td>0.33</td>
</tr>
<tr>
<td>STAGGER</td>
<td>1.005</td>
<td>1.93</td>
<td>0.28</td>
<td>0.31</td>
</tr>
<tr>
<td>ZEUS</td>
<td>1.037</td>
<td>0.76</td>
<td>0.16</td>
<td>0.27</td>
</tr>
</tbody>
</table>

|   | a Mean specific kinetic energy density at $t = 4t_{\text{dyn}}$ normalized by the reference solution. |
|   | b A proxy for the mean dissipation rate of specific kinetic energy at $t = 0.4t_{\text{dyn}}$. |
|   | c Effective spectral bandwidth for the velocity. |
|   | d Ratio of dilatational-to-solenoidal power averaged over $k/k_{\text{min}} > 100$ at $t = 4t_{\text{dyn}}$. |
Grid codes maintain different levels of magnetic energy within a factor of $\sim 1.4$.

FLASH and PPML preserve the highest levels of $E_M$.

STAGGER shows the lowest levels of $E_M$. 
Supersonic MHD turbulence decay

Evolution of the rms Alfvénic Mach number, $512^3$

- $M_A$ is similar to $E_K/E_M$ $\implies$ codes disagree.
- Alfvénic Mach number decreases from $\sim 4.5$ through $\sim 2$ as the turbulence decays.


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Supersonic MHD turbulence decay

A relative measure of the magnetic Reynolds number, $512^3$

- **PPML** and **FLASH** show the highest $Rm_{\text{eff}}$.
- **STAGGER**, **LL-MHD**, and **ENZO** show the lowest $Rm_{\text{eff}}$.

Supersonic MHD turbulence decay

Magnetic energy spectra: $512^3$, $t = 0.4 t_{\text{dyn}}$

- **PPML** and **FLASH** preserve more power at high wavenumbers.
- **STAGGER** is the most dissipative.
Magnetic energy spectra: $512^3$, $t = 1.2t_{\text{dyn}}$

- **PPML** and **FLASH** preserve more power at high wavenumbers.
- **STAGGER** is the most dissipative.
Supersonic MHD turbulence decay

Magnetic energy spectra: $512^3$, $t = 4.0 t_{\text{dyn}}$

- **PPML** and **FLASH** preserve more power at high wavenumbers.
- **STAGGER** is the most dissipative.

Supersonic MHD turbulence decay

Compensated magnetic energy spectra: $256^3$, $t = 0.4t_{\text{dyn}}$

- **FLASH** and **PPML** have the largest effective bandwidth.
- Effective bandwidth of **FLASH** is $\sim 3.2$ times larger than that of **STAGGER**.
## Selected numeric values for the B-field

<table>
<thead>
<tr>
<th>Code</th>
<th>$E_M / E_{M,\text{ref}}$ $^a$</th>
<th>$J^2$ $^b$</th>
<th>B-bandwidth $^c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ENZO</td>
<td>0.78</td>
<td>0.92</td>
<td>0.07</td>
</tr>
<tr>
<td>FLASH</td>
<td>0.94</td>
<td>1.38</td>
<td>0.20</td>
</tr>
<tr>
<td>KT-MHD</td>
<td>0.85</td>
<td>1.30</td>
<td>0.13</td>
</tr>
<tr>
<td>LL-MHD</td>
<td>0.81</td>
<td>0.80</td>
<td>0.10</td>
</tr>
<tr>
<td>PLUTO</td>
<td>0.92</td>
<td>1.14</td>
<td>0.12</td>
</tr>
<tr>
<td>PPML</td>
<td>0.92</td>
<td>1.46</td>
<td>0.20</td>
</tr>
<tr>
<td>RAMSES</td>
<td>0.87</td>
<td>1.18</td>
<td>0.09</td>
</tr>
<tr>
<td>STAGGER</td>
<td>0.70</td>
<td>0.79</td>
<td>0.07</td>
</tr>
<tr>
<td>ZEUS</td>
<td>0.83</td>
<td>1.01</td>
<td>0.10</td>
</tr>
</tbody>
</table>

$^a$ Mean magnetic energy density at $t = 4t_{\text{dyn}}$ normalized by the reference solution.

$^b$ A proxy for the mean dissipation rate of magnetic energy at $t = 0.4t_{\text{dyn}}$.

$^c$ Effective spectral bandwidth for the magnetic field.
Higher-order-accurate methods do better than lower-order-accurate methods in general.

The spatial order of accuracy is the primary determinant of velocity spectral bandwidth and effective Reynolds number.

Higher spatial order correlates with higher spectral bandwidth. The sixth-order code STAGGER is superior to the third-order codes PPML, PLUTO, KT-MHD and FLASH, which are superior to the second-order codes ZEUS, LL-MHD, and ENZO.

Codes with high velocity spectral bandwidth do not necessarily have high magnetic spectral bandwidth. For example, STAGGER has the highest velocity ESB but the lowest magnetic ESB.
The magnetic ESB is sensitive to the spatial order of accuracy of the electric field computation, and is higher in methods that interpolate on characteristic variables as opposed to primitive variables.

Artificial viscosity applied on shocks reduces the velocity ESB relative to Godunov methods that do not use artificial viscosity.

Explicit divergence cleaning reduces the magnetic spectral bandwidth relative to CT methods that preserve $\nabla \cdot \mathbf{B} = 0$ exactly.

Other algorithmic choices such as finite-difference versus finite-volume discretization, directionally split versus unsplit updates of the conservations laws, and order of accuracy of the time integration are less well correlated with the performance metrics, and therefore appear to be less important in predicting a method’s behavior on MHD turbulence.
The best performers overall are PPML, FLASH, PLUTO, and RAMSES based on velocity and magnetic Reynolds numbers and ESBs.

The highest fluid Reynolds number was obtained with the STAGGER code.

The highest magnetic Prandtl number was obtained with the FLASH code.

FLASH is somewhat more diffusive on the hydro part than magnetically, and the reverse is true for the RAMSES code.

The dilatational velocity power spectra of KT-MHD, and ENZO exhibit problematic behavior on small scales that is likely related to the ways these codes maintain $\nabla \cdot \mathbf{B} = 0$. 

Potentially interesting follow-up studies:

☞ SPH-MHD [Gaburov & Nitadori, 2010]

☞ Higher order MHD? [S. Li, 2010; Rembiasz et al., 2011]

☞ New FLASH MHD? [Waagan et al., 2011]

☞ New AREPO MHD? [Pakmor, Bauer & Springel, 2011]

☞ ATHENA?

☞ . . .

We can provide the initial conditions and help with data analysis!