Outline

• 3D direct energy cascade in supersonic turbulence
  – New exact scaling relations
  – Verification in numerical experiments

• Quasi-2D inverse energy cascade in galactic disks (?)
  – Theoretical concepts, effects of stratification and rotation
  – Predictions: self-regulation of GMC structures (?)
How could we simulate star formation ab initio?

Zooming-in from the disk scale height ($h \sim 100$ pc) down to rotationally supported disks ($\sim 100$ AU)
I. Supersonic Turbulence in 3D
Density field in a thin slice

Mach 6, grid resolution $2048^3$, 5 flow crossing times [Kritsuk et al. 2009]

Feedback & Energy Cascades
Mach 6, grid resolution $2048^3$, 5 flow crossing times [Kritsuk et al. 2009]
What is supersonic turbulence about: div() or curl()?

A snapshot of $\nabla \times \mathbf{u}$ at Mach 6, grid resolution $1024^3$ [Kritsuk et al. 2007]
Third-order structure functions of velocity do not scale linearly:

\[ S_3(u, \ell) \propto \ell^{1.3} \]

- $1024^3$ model of isothermal HD turbulence, Mach 6, no self-gravity [Kritsuk et al. 2007].
- Density-weighted velocity: $v = \rho^{1/3} u$. Total energy is conserved: $E = \langle \rho u^2 / 2 + c_s^2 \rho \ln \rho \rangle$.
- Linear scaling: $S_3(v, \ell) \propto \ell^1$ independent of the Mach number.

$\nu \equiv \rho^{1/3} u$ shows "universal" behavior?
Larson’s scaling relations

$^{12}$CO J=1-0 data compilation for Galactic molecular clouds [Hennebelle & Falgarone 2012]
Beyond dimensional arguments

- Falkovich, Fouxon & Oz (2010, JFM)
  *New relations for correlation functions in Navier-Stokes turbulence*

- **Galtier & Banerjee (2011, PRL)**
  *Exact relation for correlation functions in compressible isothermal turbulence*

- Wagner, Falkovich, Kritsuk & Norman (2012, JFM)
  *Flux correlations in supersonic isothermal turbulence*

- **Aluie (2013, Phys. D)**
  *Scale decomposition in compressible turbulence*

- Banerjee & Galtier (2013, PRE)
  *Exact relation with two-point correlation functions and phenomenological approach for compressible magnetohydrodynamic turbulence*

- **Kritsuk, Wagner & Norman (2013, JFM Rapids)**
  *Energy cascade and scaling in supersonic isothermal turbulence*

- Banerjee & Galtier (2014, JFM)
  *A Kolmogorov-like exact relation for compressible polytropic turbulence*
Compressible N-S equations, isothermal EOS

\[ \partial_t \rho + \nabla \cdot (\rho u) = 0, \]  
\[ \partial_t (\rho u) + \nabla \cdot (\rho uu) + \nabla p = \eta \Delta u + \frac{\eta}{3} \nabla (\nabla \cdot u) + f \]  

\[ p = c_s^2 \rho, \quad \eta > 0, \quad f(x, t) \] — random force.

Total energy density is an ideal invariant: \( E \equiv \langle \rho u^2/2 + \rho e \rangle \), where \( e = c_s^2 \ln(\rho / \rho_0) \).

The energy balance equation: injection versus dissipation

\[ \partial_t E = \langle \epsilon \rangle - \eta \langle \omega^2 + 4d^2/3 \rangle \]  
\( \epsilon = u \cdot f \) — energy injection rate, \( \omega = \nabla \times u \) — vorticity,  
\( d = \nabla \cdot u \) — dilatation, and \( \langle \ldots \rangle \) — ensemble average.

Dissipative anomaly in 3D: \( \langle \epsilon \rangle = \eta \langle \omega^2 + 4d^2/3 \rangle = O(1) \) even if \( \eta \ll 1 \)
In a statistical steady state at $Re \gg 1$, assuming isotropy,

$$Q(r) + F_{\parallel}(r) = -\frac{4}{3} \varepsilon r,$$

where

$$F_{\parallel}(r) \equiv F \cdot r / r = \langle [\delta(\rho u) \cdot \delta u + 2\delta \rho \delta e] \delta u_{\parallel} + \tilde{\delta} e \delta (\rho u_{\parallel}) \rangle$$

$$Q(r) \equiv \frac{1}{r^2} \int_0^r S(r) r^2 dr$$

$$S(r) = \langle [\delta (d \rho u) - \tilde{\delta} d \delta (\rho u)] \cdot \delta u + 2[\delta (d \rho) - \tilde{\delta} d \delta \rho] \delta e + \delta d \delta p - 2 dp \rangle$$

$$\varepsilon = \langle \rho u' \cdot a + \rho' u' \cdot a \rangle / 2$$

[Galtier & Banerjee, 2011; Kritsuk et al. 2013a]
The new relation holds reasonably well; \( \text{sign}(F_\parallel) = -\text{sign}(S); |F_\parallel|/S \approx 3.2 \)

Direct energy cascade with an effective sink due to compressibility.

Both \( F_\parallel (r) \) and \( Q(r) \) scale \( \sim \) linearly with \( r \)
In a statistical steady state at $Re \gg 1$, assuming isotropy,

$$Q(r) + F_\parallel (r) = -\frac{4}{3} \varepsilon r,$$

where

$$F_\parallel (r) \equiv F \cdot r / r = \langle [\delta (\rho u) \cdot \delta u + 2 \delta \rho \delta e] \delta u_\parallel + \tilde{\delta} e \delta (\rho u_\parallel) \rangle$$

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$$\varepsilon = \langle \rho u' \cdot a + \rho' u' \cdot a \rangle / 2$$

[Galtier & Banerjee, 2011; Kritsuk et al. 2013a]
\[ F_{\parallel}(r) = \langle [ \delta(\rho u) \cdot \delta u] \, \delta u_{\parallel} \rangle \]; \[ \mathcal{I}(r) = \langle [ \delta(d \rho u) - \delta \delta(\rho u)] \cdot \delta u \rangle \]; \[ \varepsilon(r) \approx \langle \rho u \cdot a \rangle = \varepsilon_0. \]
• Ignoring subdominant terms representing fluctuations of pressure $p$ and compressive energy $e$, we get

$$\mathcal{Q}(r) + \langle [\delta(\rho u) \cdot \delta u] \delta u_\parallel \rangle = -\frac{4}{3} \mathcal{C} \epsilon_0 r$$

• As $\mathcal{Q}(r) \propto r$, it can be incorporated in $\epsilon_{\text{eff}}$

$$\langle [\delta(\rho u) \cdot \delta u] \delta u_\parallel \rangle = -\frac{4}{3} \epsilon_{\text{eff}} r$$

• Compare with a primitive version of the 4/5 law for incompressible turbulence

$$\langle (\delta u)^2 \delta u_\parallel \rangle = -\frac{4}{3} \bar{\epsilon} r,$$

where $\bar{\epsilon} = \epsilon / \rho_0$

• The new relation allows to derive Larson’s laws from first principles, see [Kritsuk et al. (2013b)].
Turbulence at Mach 17

Isothermal fluid, grid resolution $4096^3$, 5 flow crossing times [Federrath 2013]

solenoidal (left) and compressive (right) random forcing
• In fully developed turbulence at high Mach numbers, solenoidal and dilatational modes are nonlinearly coupled [Moyal (1952); Kovasznay (1953)].

• A tendency toward energy equipartition between the modes has been demonstrated 60 years ago [Kraichnan (1955)], see also [Goldreich & Kumar 1988].

• The steep spectrum of $\rho^{1/3} u$ measured by Federrath (2013) is most likely due to a combination of nonequilibrium nature of the compressive forcing and limited grid resolution.

• Conjecture $|Q(r)| > |F(r)|$ used by Galtier & Banerjee (2011) to derive the $-19/9$ slope is not supported by numerical simulations.
What if we add magnetic field?

Elsässer fields: $z^\pm = v \pm v_A$

Alfvén velocity: $v_A = b/\sqrt{4\pi \rho}$

Total energy density is an ideal invariant:

$$E = \langle \rho (v^2 + v_A^2)/2 + \rho e \rangle$$

$$-2\varepsilon = \frac{1}{2} \nabla_r \cdot \left[ \left( \frac{1}{2} \delta (\rho z^-) \cdot \delta z^- + \delta \rho \delta e \right) \delta z^+ + \left( \frac{1}{2} \delta (\rho z^+) \cdot \delta z^+ + \delta \rho \delta e \right) \delta z^- + \delta \left( e + \frac{v_A^2}{2} \right) \delta (\rho z^- + \rho z^+) \right]$$

$$- \frac{1}{8} \left\{ \frac{1}{\beta'} \nabla' \cdot (\rho' z^+ e') + \frac{1}{\beta} \nabla \cdot (\rho z^+ e) + \frac{1}{\beta'} \nabla' \cdot (\rho z^- e') + \frac{1}{\beta} \nabla \cdot (\rho' z^- e) \right\}$$

$$+ \left\langle (\nabla \cdot v R_E' - E' - \frac{\delta \rho}{2} (v_A' \cdot v_A) + \frac{P' - P}{2}) \right\rangle + \left\langle (\nabla' \cdot v') R_E - E - \frac{\delta \rho}{2} (v_A \cdot v_A') + \frac{P_M - P}{2} \right\rangle$$

$$+ \langle (\nabla \cdot v_A) [R_H - R_H' + H' - \overline{\delta \rho (v' \cdot v_A)]} + \langle (\nabla' \cdot v_A') [R_H' - R_H + H - \overline{\delta \rho (v \cdot v_A')}] \rangle ,$$

Compressible cross-helicity density:

$H = \rho v \cdot v_A$

Two-point correlations associated with the total energy and cross-helicity:

$$R_E = \rho (v \cdot v' + v_A \cdot v_A')/2 + \rho e';$$

$$R_H = \rho (v \cdot v_A' + v_A \cdot v')/2$$

[Banerjee & Galtier, 2013]
New analytical scaling relation for supersonic isothermal turbulence.

Linear scaling of \( S_3(v, r) \equiv \langle |\delta(\rho^{1/3}u)|^3 \rangle \) with \( r \) previously seen in numerical experiments is dimensionally consistent with the analytical result.

Scaling range of \( \langle [\delta(\rho u) \cdot \delta u] \delta u_{\parallel} \rangle \propto r \) is more extended compared to that of \( S_3(v, r) \), indicating that the ‘symmetric’ density weighting in \( S_3(v, r) \) only approximately reflects the N-S dynamics.

A number of various statistics obtained in numerical models of supersonic turbulence (e.g. density PDF, mass–size and velocity–size correlations) agree with observational measurements in molecular clouds.

Compressive and solenoidal modes in fully developed supersonic turbulence approach a state of equilibrium. Hence, the linear scaling of \( \langle [\delta(\rho u) \cdot \delta u] \delta u_{\parallel} \rangle \) is universal in the inertial range at high Mach numbers (i.e. it does not depend on how the energy is injected on large scales).
II. Quasi-2D Turbulence in Disks?
The break is at 100 – 200 pc, interpreted as the LOS thickness of the LMC disk.

LMC with *Spitzer* at 160 µm, see also similar breaks in M33 [Combes et al. 2012] [Block et al. 2014; also Elmegreen et al. 2001, Padoan et al. 2001]
The break is at $\sim 150$ pc; the scale height of the disk $\sim 200$ pc

$4 \times 7$ kpc snapshot from AMR simulation with Ramses ($\Delta_{\text{eff}} = 0.8$ pc)

[Bournaud et al. 2010]
Power spectra for the three velocity components; $P(V_z)$ flattens at scales $\gtrsim 150$ pc.

The large-scale gas motions in the disk are quasi-2D; small-scale – globally isotropic

[Bournaud et al. 2010]
Vortices of various sizes, intensity, and spin directions (global rotation subtracted)

Right: Vorticity map for large-scale in-plane velocity components

Disk scale height should be resolved with at least $(100 - 200)\Delta$!

[Bournaud et al. 2010]
Fluid/gravitational instabilities in galactic disks

Spiral potential, gas self-gravity, heating and cooling, chemistry

TVD MUSCL, Cartesian grid $20 \times 4096^2$ [Khoperskov et al. 2013]
Horizontal slices of vertical vorticity $\omega_z$ for two flow configurations

**Left:** $R = 0$, $S = 3/16$, $\epsilon_v/\epsilon_I = 0.71$ and **Right:** $R = 1.5$, $S = 4$, $\epsilon_v/\epsilon_I = 0.66$

Control parameters: $R = \frac{\Omega}{(k^2f\epsilon_I)^{1/3}} \sim (Ro)^{-1}$ and $S = L_z/\ell_f$

- Symmetry breaking induced by rotation
- Strong rotation and tight vertical confinement promote the inverse cascade

[Deusebio et al. 2014], see also [Pouquet & Marino 2013]
GMC rotation in models with strong feedback

Spiral potential, gas self-gravity, heating and cooling, feedback

SPH, 8M particles [Dobbs & Pringle 2013]

- 39%–44% of GMCs show retrograde rotation; most massive GMCs tend to be prograde
- Consistent with observations [Blitz 1993; Phillips 1999; Rosolowsky et al. 2003; Imara & Blitz 2011; Imara et al. 2011]
Is the inverse energy cascade (coupled with gravity) at work in ISM of disk-like galaxies?

What determines rotational properties of GMCs?

How star formation self-regulates across scales?

What feeds interstellar turbulence in galactic disks?

At what scale most of the energy is injected?

Where does this energy go?

What if $\epsilon_I \neq \epsilon_\nu$?

Why could not I come to KITP in May?
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